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Equal-speed pursuit and evasion on manifolds

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Abstract
Two sides play a game of equal-speed pursuit and capture on a complete Riemannian manifold of dimension at least two, with or without boundary. When capture is defined strictly, the evader can always escape capture indefinitely, although the pursuer can generally close within arbitrary distance in finite time.

Keywords Pursuit and evasion · Riemannian geometry · Differential games

Mathematics Subject Classification 49N75

1 Introduction

The lion and man problem, first posed by Rado in the late 1930s, is a classical game of pursuit and capture in which two players move about a circular arena in $\mathbb{R}^2$ with equal maximum speeds, one trying to reach the same position as the other. The final word on this problem was delivered by Besicovitch in 1952 and publicized by Littlewood in 1953: the evader always wins, regardless of the strategy of the pursuer (Littlewood 1986).

The temptation to modify and expand this problem has proved irresistible. Additional pursuers (Chodun 1989; Hagedorn and Breakwell 1976; Von Moll et al. 2019) and restrictions on the evader's movement (Croft 1964; Kuchkarov 2010) have been introduced to increase the possibility of capture, while decoys (Lewin 1973) and blockers (Fisac and Sastry 2015) have been posited to decrease that possibility. The assumption of equal maximal velocities has been loosened (Flynn 1974), as has the requirement of strict capture (Lewin 1986; Alonso et al. 1992; Yufereva 2018; Von Moll et al. 2022). Discrete-time analogues have been formulated and analyzed (Kopparty and Ravishankar 2005; Mycielski 1988; Sgall 2001), as have pursuit games on graphs (Adler et al. 2003) and less-structured spaces Alexander et al. (2006);

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Barmak (2018). Finally, a number of authors have considered the possibility that one or both of the players possesses less-than-complete information about the game state (Chernousko and Melikyan 1975; Hexner 1979; Yavin 1986).

In this paper, we consider Rado's lion and man problem (continuous time, single pursuer with equal velocity, strict capture condition) on an arbitrary Riemannian manifold, a generalized environment in which the flavor of the original formulation is nonetheless maintained.

A. Melikyan and others have considered related pursuit problems on certain two-dimensional manifolds, (Hovakimyan and Melikyan 2000; Melikyan 1998, 2007), though their focus has always been on unequal-speed problems where capture is assured (faster pursuer) or best approach is well-defined (faster evader). As we shall see, the equal-speed problem represents a boundary case where more specialized techniques are needed.

The structure of the paper is as follows. Sections 2 and 3 establish notation and terminology, borrowing from both Riemannian geometry and the theory of differential games. Here we discuss the game-theoretic notion of a *payoff function* and see why it cannot be easily applied to general equal-speed pursuit problems.

Section 4 describes two historically-important pursuit strategies (radial and Besicovitch) in the plane, while Sects. 5 and 6 extend the ideas of those strategies to general Riemannian manifolds. We shall see that such strategies can be profitably compared to corresponding ones in two-dimensional, constant-curvature hyperbolic and spherical spaces. Such comparisons lead to our first major result, Theorem 1, which establishes that under very weak conditions, an equal-speed pursuer can never catch its prey on a Riemannian manifold.

Section 7 considers the natural question of nearest approach, leading to the paper's second major result: when following the successful strategy outlined in Theorem 1, the evader cannot prevent an equal-speed pursuer from closing to within any arbitrary distance (Theorem 2).

Section 8 considers these new theorems in light of the seemingly contradictory result of Bollobás, et. al (Bollobás et al. 2012), while Sect. 9 offers brief closing thoughts.

2 Preliminaries

In the spirit of the original lion and man problem, we assume that each player moves along a piecewise-differentiable path in a complete Reimannian manifold M, which may perhaps include a boundary ∂M. Formally,

$$\begin{aligned} \gamma_P(t) &\in M, \qquad \gamma_P'(t) = f_P(\gamma_P, \gamma_E), \\ \gamma_E(t) &\in M, \qquad \gamma_E'(t) = f_E(\gamma_P, \gamma_E), \end{aligned} \tag{1}$$

where the functions f_P and f_E in these state equations define the players' instantaneous movements. The speeds of the players are bounded by a common value v, which may be taken to be identically 1 without loss of generality, an assumption that we will make throughout.

Both pursuer and evader have instantaneous and complete knowledge of the game state but no prescience of the opponent's strategy. The sets of all possible evader and pursuer paths for specified starting points E_0 and P_0, respectively, are thus given by

$$\Gamma_E = \{\gamma_E : [0,\infty) \to M : \gamma_E(0) = E_0,\ d(\gamma_E(t_1), \gamma_E(t_2)) \leq |t_2 - t_1| \ \ \forall t_1, t_2 \geq 0\}$$
$$\Gamma_P = \{\gamma_P : [0,\infty) \to M : \gamma_P(0) = P_0,\ d(\gamma_P(t_1), \gamma_P(t_2)) \leq |t_2 - t_1| \ \ \forall t_1, t_2 \geq 0\}, \tag{2}$$

namely the Lipschitz paths in M with the given starting points. Here and elsewhere in this paper, the distance function $d(x, y)$ on M represents the solution to the variational problem

$$d(x,y) = \min_{\xi} \int_{\xi} \sqrt{\langle G(\xi)\xi', \xi' \rangle}\, dt \tag{3}$$

where G is the usual nonsingular, positive-definite Riemannian metric tensor, the angled braces represent the standard scalar product, and the minimum ranges over all piecewise-continuous paths $\xi(t)$ in M connecting x to y. At least one *geodesic* (locally distance-minimizing path) from x to y realizing this minimum is always guaranteed to exist on a complete manifold (Do Carmo 1992).

More generally, the integral in Eq. 3 defines the *length* L of any curve ξ connecting x and y.

Adapting terminology from (Bollobás et al. 2012), we define a *non-anticipative evader strategy* to be a map $\varrho_E : \Gamma_P \to \Gamma_E$ such that for any two pursuer paths γ_P and $\widetilde{\gamma}_P$,

$$\gamma_P(t) = \widetilde{\gamma}_P(t) \text{ for } t \in [0, t_0] \implies \varrho_E(\gamma_P)(t) = \varrho_E(\widetilde{\gamma}_P)(t) \text{ for } t \in [0, t_0]. \tag{4}$$

while a *non-anticipative pursuer strategy* is a map $\varrho_P : \Gamma_E \to \Gamma_P$ such that

$$\gamma_E(t) = \widetilde{\gamma}_E(t) \text{ for } t \in [0, t_0] \implies \varrho_P(\gamma_E)(t) = \varrho_P(\widetilde{\gamma}_E)(t) \text{ for } t \in [0, t_0], \tag{5}$$

for any evader paths γ_E and $\widetilde{\gamma}_E$. In other words, if two evader (or pursuer) paths coincide through time t_0, then the corresponding pursuer (or evader) paths will coincide as well. This encodes both the deterministic quality of the game as well as the stipulation that neither player has foreknowledge of the other's actions.

Capture is said to occur if $\gamma_P(T) = \gamma_E(T)$ for some finite time T. The pursuer is said to *win* if such a T exists, while the evader is said to win if it does not.

3 Differential games of pursuit and capture

A game-theoretic approach has often been applied to questions of pursuit and capture. Isaac's text is seminal (Isaacs 1965); see (Weintraub et al. 2020) for a more recent survey of work done using this perspective.

A defining feature of the game-theoretic approach is the positing of a payoff function which takes the paths γ_P and γ_E as arguments. If capture occurs, for instance in the case of a faster pursuer, this takes the form

$$J_1(\gamma_P, \gamma_E) = \min_{t>0}\{t : \gamma_P(t) = \gamma_E(t)\}, \tag{6}$$

while if capture does not occur, for instance in the case of a faster evader, it is instead

$$J_2(\gamma_P, \gamma_E) = \min_{t\geq 0} d(\gamma_P(t), \gamma_E(t)), \tag{7}$$

indicating a game of approach (Harutunian 1998).

In each of the above cases, the pursuer seeks to minimize J_i while the evader seeks to maximize it, making both games zero-sum. The *value* of the game is then defined as a minimax of the appropriate payoff function J_i.

$$\min_{\gamma_P\in\Gamma_P}\max_{\gamma_E\in\Gamma_E} J_i = \max_{\gamma_E\in\Gamma_E}\min_{\gamma_P\in\Gamma_P} J_i \tag{8}$$

For many configurations on a wide variety of manifolds, best play for both players always consists of moving along the unique minimal geodesic connecting $\gamma_P(t)$ and $\gamma_E(t)$, a direct consequence of the first variational formula. Considering the fast-evader case on two-dimensional manifolds, for instance, Melikyan and Ovakimyan Melikyan and Ovakimyan (1993) give the gradients of the distance function d (as defined in section 2) when first γ_E and then γ_P are held constant as

$$d_P(\gamma_P, \gamma_E) = G(\gamma_P)a, \qquad d_E(\gamma_P, \gamma_E) = G(\gamma_E)b, \tag{9}$$

where a and b are the outward unit tangent vectors to the geodesic segment ξ connecting γ_P to γ_E,

$$a = -\xi'(0)/|\xi'(0)|, \qquad b = \xi'(d)/|\xi'(d)|, \qquad d = d(\gamma_P, \gamma_E). \tag{10}$$

Then the equality

$$\frac{d}{dt}d(\gamma_P, \gamma_E) = \langle d_P(\gamma_P, \gamma_E), \gamma_P'\rangle + \langle d_E(\gamma_P, \gamma_E), \gamma_E'\rangle \tag{11}$$

implies that optimal movement for each player must be parallel to a and b. Starting configurations (P_0, E_0) where this strategy is well-defined for both players and all times $t \geq 0$ are said to make up the *primary domain* of the configuration space. Many of the references cited in this section have focused on identifying and analyzing the *secondary domain* of specific manifolds, that is, the sets of configurations for which the direct approach is more ambiguous.

Such game-theoretic tools do not carry over well to the case of equal-speed pursuit and evasion. As we shall see in sections 5-7, the minimax of the payoff function J_1 (time to capture) is always infinite while the minimax of J_2 (minimum distance) is usually zero. Thus it is necessary to either posit a less-standard payoff function, for instance by considering non-zero capture radius, or adopt more specialized techniques. We opt for the latter approach.

4 Strategy-specific approaches

A more geometric approach is profitable in a wide variety of circumstances, including but not limited to equal-speed pursuit and evasion. See Nahin (2012) for an overview of some of the work that has been done from this perspective.

Several particular strategies are worth mentioning at the outset in light of their historical importance, their relevance to the present problem, and their overall utility.

First and foremost among these are the natural strategies of *pure pursuit* and *pure evasion*, where one or both of the parties moves at maximum speed along the common geodesic connecting them, the one toward its opponent and the other away. In the absence of other constraints, these strategies are optimal for both players, as indicated in Sect. 3. In particular, in $\mathbb{R}^n$ neither player can improve on them, regardless of their relative maximal speeds. The geometry of the playing arena thus becomes a significant complicating factor in the problem.

In the equal-speed case, the strategy of pure pursuit is ineffective even in compact arenas. In the original lion and man problem in the unit disk, for instance, an evader circling around the boundary S^1 will never be intercepted by such a pursuer (Hathaway et al. 1921). This fact illustrates a general problem with the strategy of pure pursuit: the chaser ultimately finds itself trailing behind the evader and unable to make up ground fast enough to effect capture, even when the evader must curve to avoid the arena boundary (Nahin 2012).

In $\mathbb{R}^2$, the situation is even more dire. Against a strategy of pure pursuit, an evader with maximum speed equal to that of its pursuer can lead that pursuer to any point in the plane that it likes without allowing it to close more than an arbitrarily small distance (Gard 2018). The chase, in that case, is worse than hopeless.

In Rado's original unit disk problem, however, the strategy of pure pursuit can be improved upon. In his presentation (Littlewood 1986), Littlewood suggests as obvious the strategy of *radial pursuit*, which initially seems to resolve the lion and man problem fully in the lion's favor. Under this strategy, the pursuer always stays on the radius connecting the evader to the center of the arena, closing inexorably along it and inevitably pressing its quarry to the wall. If the evader circles around the boundary of the arena as suggested above, the pursuer traces a semicircular path of unit diameter and effects capture at $t = \pi/2$, as shown in Fig. 1.

Although this radial strategy is in a certain sense optimal for the pursuer, as shown in Sect. 7, the evader can do better than simply orbiting around the boundary of the arena. Under the *Besicovitch strategy*, it begins at an interior point of the arena, then spirals asymptotically towards its boundary, somehow keeping the

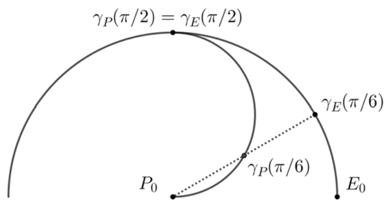

Fig. 1 The radial strategy in the unit disk

pursuer at bay forever. This strategy is detailed in the next section, where it will be shown that the restriction to a unit disk in $\mathbb{R}^2$ is unnecessary.

5 Evasion in Riemannian manifolds: radial pursuit

The evader strategy described below assumes an initial state in which E_0 lies within a geodesic ball centered at P_0, an assumption validated by the following standard result.

Theorem *At any point x of a Riemannian manifold M, there exists an $\epsilon > 0$ such that the exponential map $exp_x : B_\epsilon(0) \subset T_xM \to M$ is a diffeomorphism of the ball of radius ϵ in $\mathbb{R}^n$ onto an open subset of M.*

The exponential map $exp_x(\boldsymbol{w})$ can be thought of as moving a distance $|\boldsymbol{w}|$ along the geodesic beginning at x and extending in the direction specified by $\boldsymbol{w}$. See for example Do Carmo (1992). Here T_xM represents the set of all tangent vectors to M at the point x, while TM (which we will encounter later) refers to the disjoint union of all such T_xM over all $x \in M$.

It follows immediately that the tangent bundle is trivial within a geodesic ball and that all geodesics are distance-minimizing there.

As there is no chance of confusion in what follows, we will also denote the image of the above exponential map B_ϵ.

If the initial game state does not satisfy $E_0 \in B_\epsilon(P_0)$ for some ϵ, the evader need only wait for the pursuer to draw closer, moving off the boundary ∂M first if necessary. In what follows, we show that the evader can remain inside $B_\epsilon(P_0)$ indefinitely without allowing capture, no matter the pursuer's strategy nor the radius ϵ of the geodesic ball.

To begin, consider the case where the pursuer follows a radial strategy akin to the one described in Sect. 4. That is, it keeps to the geodesic segment connecting $\gamma_E(t)$ and P_0, gradually closing on its target in an attempt to drive it toward the boundary of $B_\epsilon(P_0)$ where it might hope to effect capture. We will later show that other strategies can only be less optimal.

The evader moves in a piecewise-geodesic fashion, the k^{th} piece γ_k beginning in a direction orthogonal to the common geodesic segment connecting $P_0, \gamma_P(t_{k-1})$, and $\gamma_E(t_{k-1})$, which we will denote σ_{k-1}. By this convention, σ_0 represents the minimal geodesic initially separating the players. For $k \geq 1$, let $l_k = L(\gamma_k)$ denote the length of the evader's k^{th} step, and let $r_k = L(\sigma_k)$ denote the distance from the evader to the ball's center at the end of the k^{th} step. Here we take $\sigma_{k-1}(0) = P_0, \sigma_{k-1}(r_{k-1}) = \gamma_E(t_{k-1}) = \gamma_k(0)$, and $\gamma_k(l_k) = \gamma_E(t_k)$ as pictured in Fig. 2.

The evader chooses $l_k = c/k$, where the constant $c > 0$ will be specified later. It is immediate that such a path continues indefinitely for any $c > 0$. We need only show (1) that c can be chosen so that the path lies entirely within $B_\epsilon(P_0)$ and (2) that $\gamma_P(t) \neq \gamma_E(t)$ at any point along any of these geodesic segments.

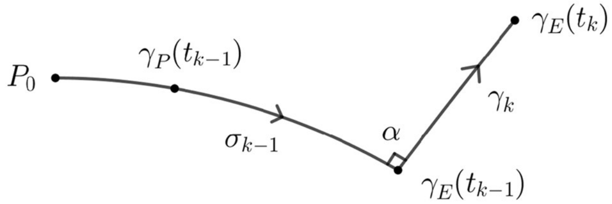

Fig. 2 The kth step in a successful evader strategy

The configuration shown in Fig. 2, defines a *geodesic hinge* (Chavel 2006) in M with angle $\alpha = \pi/2$ at the point $\gamma_E(t_{k-1})$. The following theorem is relevant.

Theorem (Alexandrov-Topogonov) *Let* $(\sigma_{k-1}, \gamma_k, \alpha)$ *be a geodesic hinge in a complete Riemannian manifold whose sectional curvature* K *is bounded below by* K_L, *and assume the geodesic* σ_{k-1} *is minimal. Let* $(\overline{\sigma_{k-1}}, \overline{\gamma_k}, \alpha)$ *be a geodesic hinge in the two-dimensional space* $\mathbb{M}^2_{K_L}$ *of constant curvature* K_L *such that* $L(\sigma_{k-1}) = L(\overline{\sigma_{k-1}}) = r_{k-1}$ *and* $L(\gamma_k) = L(\overline{\gamma_k}) = l_k$. *Then*

$$d(\sigma_{k-1}(0)), \gamma_k(l_k)) \leq d(\overline{\sigma_{k-1}}(0), \overline{\gamma_k}(l_k)) \tag{12}$$

In other words, the distance between the endpoints of the hinge are closer together than they would be in a model space $\mathbb{M}^2_{K_L}$ *with lesser curvature.*

Similarly, if K is bounded above by K_U, then

$$d(\sigma_{k-1}(0)), \gamma_k(l_k)) \geq d(\overline{\sigma_{k-1}}(0), \overline{\gamma_k}(l_k)). \tag{13}$$

provided $l_k \leq \pi/\sqrt{K_U}$ (Berger 2003).

The sectional curvature K of a Riemannian manifold M is a continuous real-valued function on the two-dimensional subspaces of TM (Bishop and Crittenden 2011). The latter, the Grassmanian bundle $G(2, \mathbb{R}^n)$, is trivial in $B_\epsilon(P_0)$, hence

$$K : \overline{B_\epsilon} \times G(2, \mathbb{R}^n) \rightarrow \mathbf{R}. \tag{14}$$

represents a continuous function on a compact manifold. As such, it attains both upper and lower bounds, K_L and K_U. In what follows, we assume $K_L < 0$ and $K_U > 0$, loosening those bounds as necessary.

Figure 3 presents the hinge from figure 2 as a geodesic right triangle together with its comparison triangle in $\mathbb{H}^2_{K_L}$, the hyperbolic plane with constant negative curvature K_L. For each $k \geq 1$, the length $\overline{r_k}$ of the hypotenuse of the comparison triangle is greater than its counterpart in M. This length can be also computed explicitly using the hyperbolic Pythagorean theorem (Brannan et al. 1999).

$$\cosh \frac{\overline{r_k}}{R} = \cosh \frac{r_{k-1}}{R} \cosh \frac{l_k}{R}, \quad R = 1/\sqrt{-K_L}. \tag{15}$$

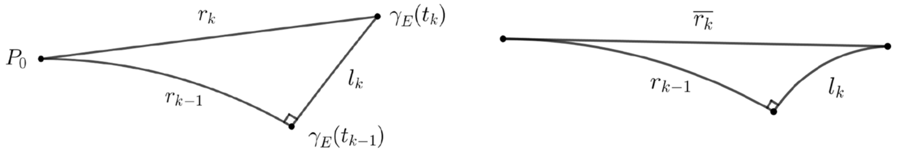

Fig. 3 Comparison triangles

Thus after the first n steps, the distance of the evader from P_0 is bounded by $\overline{r_n}$, where

$$\cosh \frac{\overline{r_n}}{R} = \cosh \frac{r_0}{R} \prod_{k=1}^{n} \cosh \frac{l_k}{R} = \cosh \frac{r_0}{R} \prod_{k=1}^{n} \cosh \frac{c}{Rk}. \tag{16}$$

To control the right-hand side, we use the fact that $\cosh x \le e^{x^2}$ when $x \ge 0$.

$$\begin{aligned} \log\cosh \frac{\overline{r_n}}{R} &= \log\cosh \frac{r_0}{R} + \sum_{k=1}^{n} \log\cosh\left(\frac{c}{Rk}\right) \\ &\le \log\cosh \frac{r_0}{R} + \sum_{k=1}^{n} \left(\frac{c}{Rk}\right)^2 \\ &\le \log\cosh \frac{r_0}{R} + \sum_{k=1}^{\infty} \left(\frac{c}{Rk}\right)^2 \\ &= \log\cosh \frac{r_0}{R} + \frac{c^2}{R^2} \cdot \frac{\pi^2}{6} \end{aligned} \tag{17}$$

And hence

$$\cosh \frac{\overline{r_n}}{R} \le \left(e^{\pi^2 c^2/6R^2}\right) \cosh \frac{r_0}{R}, \tag{18}$$

where the evader chooses $c > 0$. Then for sufficiently small c, $r_n \le \overline{r_n} < r_0 + \delta$ for any fixed $\delta > 0$. Not only does the evader not leave the ball B_ϵ, it need not move any farther away from its center, P_0, than it might wish.

Just as the above comparison with $\mathbb{H}^2_{K_L}$ proves that the evader need never leave B_ϵ, the following comparison with $\mathbb{S}^2_{K_U}$ shows that the pursuer can never effect capture.

Suppose by way of contradiction that the pursuer is able to capture the evader on one of these steps, arriving at a point Q along some γ_k after traveling the same distance $\Delta t \le l_k$ as its quarry. Let $d_{k-1} = d(\gamma_P(t_{k-1}), \gamma_E(t_{k-1}))$ denote the the initial distance between players at the start of the k^{th} step and $d_{\Delta t} = d(\gamma_P(t_{k-1}), Q)$ be the distance between the pursuer's initial position and the point of capture, a distance which need not necessarily correspond to any part of the path γ_P. Then Δt is bounded below by $d_{\Delta t}$, which is in turn bounded below by the hypotenuse of the corresponding right triangle in $\mathbb{S}^2_{K_U}$, the two-sphere with curvature K_U, again by the Alexandrov-Toponogov theorem, provided that $\Delta t < \pi/\sqrt{K_U}$. Since the step size $l_k = c/k$ bounds Δt and is controlled by the evader through the constant c, this last condition can be assumed to be satisfied.

The situation is illustrated in Fig. 4, where $\Delta t \geq d_{\Delta t} \geq \overline{d_{\Delta t}}$. Here the left rightmost triangle lies in the model space $\mathbb{S}^2_{K_U}$ while the others lie in M.

Applying the spherical Pythagorean theorem (Brannan et al. 1999) to the comparison triangle,

$$\begin{aligned} \cos\left(\frac{\overline{d_{\Delta t}}}{R}\right) &= \cos\left(\frac{\Delta t}{R}\right)\cos\left(\frac{d_{k-1}}{R}\right), \qquad R = \frac{1}{\sqrt{K_U}} \\ &\leq \cos\left(\frac{\Delta t}{R}\right) \end{aligned} \tag{19}$$

In order for this to be true while $0 < \overline{d_{\Delta t}} \leq \Delta t$, it must be the case that $\Delta t/R > \pi$, that is, $\Delta t > \pi K_U$. Recall that on the k^{th} step, $\Delta t \leq l_k = c/k$ where c is controlled by the evader. Thus it need only choose $c < \pi K_U$ to ensure that capture never occurs on any segment γ_k.

We have proved the following.

Lemma 1 *In a complete Riemannian manifold, the radial strategy of pursuit fails to drive the evader out of any open geodesic ball that includes the evader.*

6 Evasion in Riemannian manifolds: non-radial pursuit

Of course, the pursuer is under no obligation to follow a radial strategy. Let us now assume that at the start of the k^{th} step, the pursuer's position $\gamma_P(t_{k-1})$ does not lie on the geodesic connecting P_0 and $\gamma_E(t_{k-1})$.

The evader's strategy is largely unchanged. On its k^{th} step, it moves along a geodesic orthogonal to the segment connecting $\gamma_P(t_{k-1})$ and $\gamma_E(t_{k-1})$, traveling a total distance of length c/k, where c is chosen as in the proof of Lemma 1. Judicious choice of the starting direction $\gamma_k'(0)$ will insure that its terminal distance r_k from P_0 is no greater than if the pursuer had chosen the radial strategy.

Let ζ_{k-1} be the geodesic segment beginning at $\gamma_P(t_{k-1})$ and extending to $\gamma_E(t_{k-1})$. Then for $k \geq 1$, $\zeta_{k-1}'(d_{k-1})$ and $\sigma_{k-1}'(r_{k-1})$ span a two-dimensional subspace of TM at $\gamma_E(t_{k-1})$. The evader chooses its direction of flight $\gamma_k'(0)$ in this subspace so that the angle $\angle(-\sigma_{k-1}'(r_{k-1}), \gamma_k'(0))$ is acute, as pictured in Fig. 5.

The points $P_0, \gamma_E(t_{k-1})$, and $\gamma_E(t_k)$ form a geodesic hinge with side lengths r_{k-1} and l_k. The enclosed angle α has measure strictly less than $\pi/2$ by construction,

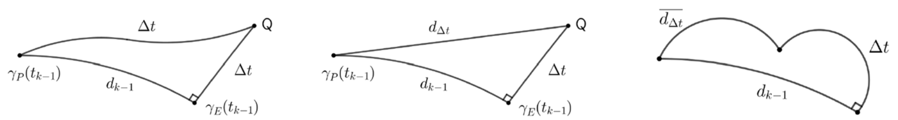

Fig. 4 The dream of capture

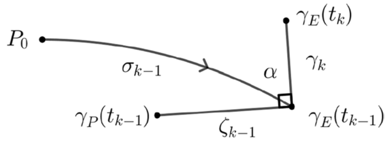

Fig. 5 The evader's kth step against a non-radial strategy

while the loose vertices P_0 and $\gamma_E(t_k)$ are separated by distance r_k. Figure 6 shows this triangle in M together with its analog in $\mathbb{H}^2_{K_L}$, where the connecting side must have length $\tilde{r}_k > r_k$ by the Alexandrov-Toponogov theorem.

We apply the hyperbolic law of cosines in $\mathbb{H}^2_{K_L}$(Brannan et al. 1999).

$$\begin{aligned}\cosh\frac{\tilde{r}_k}{R} &= \cosh\frac{r_{k-1}}{R}\cosh\frac{l_k}{R} - \sinh\frac{r_{k-1}}{R}\sinh\frac{l_k}{R}\cos\alpha \\ &\leq \cosh\frac{r_{k-1}}{R}\cosh\frac{l_k}{R} \\ &= \cosh\frac{\overline{r_k}}{R},\end{aligned} \tag{20}$$

where $\overline{r_k}$ is the length of the hypotenuse of the corresponding geodesic right triangle in $\mathbb{H}^2_{K_L}$, as in the proof of Lemma 1. Thus, $r_k \leq \tilde{r}_k \leq \overline{r_k}$, and we have proved the following.

Theorem 1 *In a continuous-time strict-capture game of pursuit and evasion in a Riemannian manifold M with $\dim(M) \geq 2$, and boundary ∂M (possibly empty), the evader always wins. That is, there exists an evader strategy for which $d(\gamma_P(t), \gamma_E(t)) > 0$ for all $t \geq 0$, regardless of the strategy chosen by the pursuer.*

7 The question of approach

Given the impossibility of capture, it is natural to consider the question of approach. Just how close can a pursuer hope to come to an evader making use of the strategy described above? More formally, we wish to investigate the possibility of minimizing the slow-pursuer payoff function

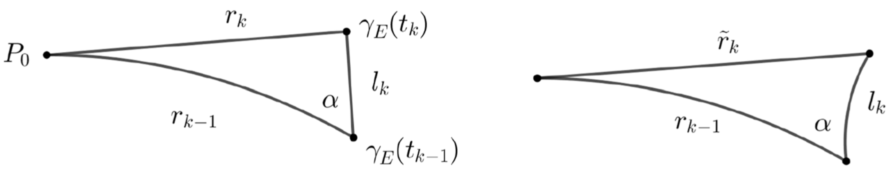

Fig. 6 Geodesic triangles in M and $\mathbb{H}^2_{K_L}$

$$J_2 = \min_{t\geq 0} d(\gamma_P(t), \gamma_E(t)) \tag{21}$$

when γ_P ranges over all possible strategies in Γ_P and γ_E follows the Besicovitch strategy. We will prove the following.

Theorem 2 *Against the Besicovitch evader strategy, the pursuer in a continuous-time strict-capture pursuit game on a Riemannian manifold with boundary can close to within any specified distance of the evader in finite time. In particular,*

$$\lim_{t\to\infty} d(\gamma_P(t), \gamma_E(t)) = 0. \tag{22}$$

When combined with Theorem 1, this implies that the payoff function J_2 does not have a well-defined minimum over Γ_P when γ_E takes the Besicovitch strategy.

Proof. Consider a radial pursuer strategy, that is, one in which the pursuer remains always on the geodesic connecting P_0 to the evader's current position $\gamma_E(t)$. Letting $r_E(t)$ and $r_P(t)$, respectively, represent the evader's and pursuer's distance to P_0, we will show that if it is not the case that $r_P \to r_E$, then $r'_P(t) > \delta$ for some fixed $\delta > 0$. This would imply that the pursuer eventually exits the arena $B_\epsilon(P_0)$, contradicting theorem 1.

Each player's position can be decomposed into radial and transverse components using induced polar coordinates,

$$\exp_{P_0} : T_{P_0}M = \mathbb{R}^+ \times S^{n-1} \to B_\epsilon(P_0). \tag{23}$$

When the pursuer follows a radial strategy, the S^{n-1} component of these coordinates is always the same for both. Let $\theta = (\theta_1, \dots, \theta_{n-1})$ represent the common angular component of their coordinates, so the players positions can be written (r_P, θ) and (r_E, θ) respectively. Then by Gauss's Lemma, the metric tensor has a block diagonal form and the tangent vector components r'_E and θ'_E are orthogonal, as are r'_P and θ'_P. In these coordinates, the unit-speed condition on $\gamma_E(t)$ becomes

$$\left(r'_E\right)^2 + r_E^2|\theta'|^2 = 1 \tag{24}$$

Hence

$$\begin{aligned} |\theta'|^2 &= \frac{1}{r_E^2}\left[1 - \left(r'_E\right)^2\right] \\ &< \frac{1}{r_E^2} \end{aligned} \tag{25}$$

This quantifies our intuition that a Besicovitch evader's motion is primarily transversal. The result then follows from consideration of the same equations for the pursuer.

$$\begin{aligned}\left(r'_P\right)^2 &= 1 - r_P^2|\theta'|^2\\ &> 1 - r_P^2\left(\frac{1}{r_E^2}\right)\\ &= 1 - \left(\frac{r_P}{r_E}\right)^2\end{aligned} \tag{26}$$

If the distance between pursuer and evader does not go to zero over time, then the ratio r_P/r_E has an upper bound strictly less than one and the quantity r'_P has a lower bound strictly greater than zero. In other words, the pursuer's distance r_P from P_0 is unbounded. This contradiction implies that $r_P \to r_E$, proving the theorem.

8 A note on continuity

In Bollobás et al. (2012), Bollobás, Leader, and Walters prove a strong result in favor of the pursuer which at first glance would seem to contradict Theorem 1.

Theorem (Bollobás/Leader/Walters) *In the lion-and-man game in the closed unit disk, the evader does not have a continuous winning strategy.*

Recall that an evader strategy is a function $\varrho_E : \Gamma_P \to \Gamma_E$ such that for any $\gamma_P, \widetilde{\gamma_P} \in \Gamma_P$,

$$\gamma_P(t) = \widetilde{\gamma_P}(t) \text{ for } t \in [0, t_0] \implies \varrho(\gamma_P)(t) = \varrho(\widetilde{\gamma_P})(t) \text{ for } t \in [0, t_0]. \tag{27}$$

A *continuous* strategy, then, is simply a map ϱ_E that is continuous in the usual topological sense. This stipulation is what allows the above result to coexist with Theorem 1.

The techniques of proof used in Bollobás et al. (2012) extend directly to the Riemannian context, so we take this opportunity to recapitulate that result in more general terms.

Suppose that in the continuous-time, strict-capture pursuit game treated in this paper, the evader has a continuous winning strategy $\varrho_E : \Gamma_P \to \Gamma_E$ on the closed arena $\overline{B_\epsilon}$, which can be obtained in the main theorem by shrinking ϵ slightly if necessary. For every point $x \in \overline{B_\epsilon}$, let $\sigma_x(t)$ be the unit-speed geodesic segment beginning at P_0 and terminating at x at time $t = T_x$. Noting that $\sigma_x \in \Gamma_P$, define a function $f(x) = \varrho_E(\sigma_x)(T_x)$, mapping $\overline{B_\epsilon}$ to itself. Then the function

$$f(y) = (\exp^{-1} \circ f \circ \exp)(y) \tag{28}$$

is a continuous map from the closed ball of radius ϵ in $\mathbb{R}^n$ to itself and must therefore have a fixed point (Hatcher 2005). Thus $f(x)$ has a fixed point as well, one for which $\varrho(P_x)(T) = P_x(T)$. In other words, there exists an evader starting point $x \in \overline{B_\epsilon}$ for which the pursuer wins. This contradiction proves the following.

Theorem 3 *In a continuous-time strict-capture pursuit game in geodesic ball contained in a Riemannian manifold, the evader cannot have a winning strategy.*

It immediately follows that the strategy outlined in Theorem 1 must not be continuous. This can be seen directly by considering a one-parameter family of configurations with $\gamma_P(t_0) = (r_P(t_0), \theta_1, \dots, \theta_{n-1})$, $t_0 \neq 0$, and $E_u(t_0) = (r_E, \theta_1 + u, \theta_2, \dots, \theta_{n-1})$ defined for u in an interval including zero. According to the strategy of theorem 1, if $u > 0$, the transversal component of the evader's flight lies in the θ_1 direction, while if $u < 0$ is lies in the $-\theta_1$ direction. Since the evader's velocity is always identically one, this implies a discontinuity in G_E, verifying that Theorem 1 is in keeping with the results of Bollobás et al.

9 Conclusion

The results of this paper nearly close the book on the equal-speed, strict-capture lion and man problem, extending the results of Besicovitch, Littlewood, and others to the general category of Riemannian manifolds with boundary. Further abstraction is of course still possible, for instance by dismissing the manifold structure entirely and allowing consideration of more general geodesic metric spaces. In particular, some (though not all) of the comparison theorems used in this paper have analogues in the CAT(k) environment.

A fundamental characteristic of the equal-speed, strict-capture pursuit problem considered here is its local nature. Throughout this paper, the evader has had the luxury of waiting until the purser entered into a sufficiently small arena before applying the local results of Riemannian geometry to its advantage. In Sect. 3, we saw that such techniques are necessary to treat equal-speed strict-capture problems, while Sects. 5–7 established that they are also quite sufficient.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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